Bound States of the Klein-Gordon Equation for Woods-Saxon Potential With Position Dependent Mass

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Abstract

The effective mass Klein-Gordon equation in one dimension for the Woods-Saxon potential is solved by using the Nikiforov-Uvarov method. Energy eigenvalues and the corresponding eigenfunctions are computed. Results are also given for the constant mass case.

Keywords: Klein-Gordon Equation, Woods-Saxon potential, position dependent mass, PT-symmetry, energy eigenvalues, eigenfunctions, Nikiforov-Uvarov method,

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I. INTRODUCTION

The solutions of the non-relativistic and relativistic wave equations have attracted attentions in recent years. Furthermore, the analytical solutions of non-linear equations have also much attention. For example, the non-linear Landau-Lifshitz equation is solved to understand the dynamics of Bose-Einstein condansates (BEC) in an optical lattice [1]. The two-component BEC with attractive interactions between atoms can be described by a non-linear Schrödinger equation (SE) called Gross-Pitaevskii equation [2].

Various methods are used to solve the Schrödinger equation based on perturbative and non-perturbative approaches such as the hypervirial-Pade summation method [3, 4], group-theoretical approach [5], Hill determinant method [6], and supersymmetric approaches [7]. The Klein-Gordon (KG) and Dirac equations are also studied for the Aharonov-Bohm (AB) potential [8], the AB plus the Dirac monopole potential [9], kink-like [10], Coulomb [11], vector plus scalar inversely linear potentials [12], PT-symmetric generalized Wood-Saxon (WS) [13], generalized Hulthen [14], and Rosen-Morse-type potentials [15]. These solutions are taken in general for constant mass [16,17]. On the other hand, position dependent mass case has also many application in different areas, such as impurities in crystals [18-20], the dependence of nuclear forces on the relative velocity of the two nucleons [21, 22], or the study of electronics properties of quantum wells and quantum dots [23], ^{3}He clusters [24], quantum liquids [25] and semiconductor heterostructures [26]. This is also to get the energy eigenvalues and eigenfunctions [27-31].

Here we intend to solve the KG-equation for the Woods-Saxon potential in the case of a exponentially mass distribution varying with position. In nuclear physics, the WS-potential is used to construct a shell model to describe the single-particle motion in a fusing system [32].

In the present work, we have obtained the energy spectrum and corresponding energy eigenfunctions by using the Nikiforov-Uvarov(NU)-method. We have also obtained the results for the constant mass case. The NU-method is developed to solve the second order linear differential equations with special orthogonal functions. The method is based on solving the equation by reducing to a generalized equation hypergeometric type [40].

The organization of this work is as follows. In Section II, we solve the KG-equation in the case of the WS-potential for the spatially dependent mass by applying the NU-method, and

give the energy eigenvalues and the corresponding eigenfunctions. Our concluding remarks are given in Section III.

II. NIKIFOROV-UVAROV METHOD AND CALCULATIONS

A. Deformed Woods-Saxon Potential

The KG-equation in one dimension for a particle reads

$$\left[\frac{d^2}{dx^2} - \frac{1}{\hbar^2 c^2}\left[m^2c^4 - (E-V)^2\right]\right]\phi(x) = 0\,, \tag{1}$$

where E is the energy of the particle, c is the velocity of the light. The Woods-Saxon potential is

$$V(x) = -\frac{V_0}{1+qe^{-\beta x}}\,, \quad (-\infty \leq x \leq \infty)\,. \tag{2}$$

is widely used in the coupled-channels calculations in heavy-ion physics. This model explains the single-particle motion during a heavy-ion collisions [32-35]. In this form of the potential, V_0 is the potential depth, q is real parameter which determines the shape of the potential, and β is a short notation, i.e. $\beta \equiv 1/a$, where a is diffuseness of the nuclear surface.

Various mass-distributions are used in the literature. These are exponential, quadratic [28], inversely-quadratic [36], trigonometric mass-distributions [37], and mass function of the form $m(r) = r^{\alpha}$, is especially used for three-dimensional problems [37, 38]. Here, we prefer to use the following position dependent mass

$$m(x) = m_0\left[1 + \frac{1}{m_0}\left(\frac{1+qe^{-\beta x}}{m_1}\right)^{-1}\right]\,, \tag{3}$$

This provides us an exact solution of the effective KG-equation. m_0 and m_1 in this distribution are two arbitrary positive parameters. m_0 will correspond to the constant mass of the particle. The mass function is finite at infinity.

Substituting Eqs. (3) and (2) into Eq. (1) we get

$$\frac{d^2\phi(x)}{dx^2}+\left[\frac{E^2}{\hbar^2c^2}-\frac{m_0^2c^4}{\hbar^2c^2}+\frac{2EV_0/\hbar^2c^2-2m_0m_1c^4/\hbar^2c^2}{(1+qe^{-\beta x})}+\frac{V_0^2/\hbar^2c^2-m_1^2c^4/\hbar^2c^2}{(1+qe^{-\beta x})^2}\right]\phi(x)=0 \quad (4)$$

To solve this equation, we use the transformation $z=(1+qe^{-\beta x})^{-1}$. By defining the following parameters

$$a_3^2=Q^2(m_0^2c^4-E^2)\,,\quad a_2^2=Q^2(2m_0m_1c^4-2EV_0)\,,\quad a_1^2=Q^2(m_1^2c^4-V_0^2)\,, \tag{5}$$

we obtain

$$\frac{d^2\phi(z)}{dz^2}+\frac{1-2z}{z-z^2}\frac{d\phi(z)}{dz}+\frac{1}{(z-z^2)^2}\left[-a_1^2z^2-a_2^2z-a_3^2\right]\phi(z)=0\,. \tag{6}$$

where $Q^2=1/\beta^2\hbar^2c^2$. Now to apply the NU-method [40], we rewrite Eq. (6) in the following form

$$\phi''(z)+\frac{\tilde{\tau}(z)}{\sigma(z)}\phi'(z)+\frac{\tilde{\sigma}(z)}{\sigma^2(z)}\phi(z)=0, \tag{7}$$

where $\sigma(z)$ and $\tilde{\sigma}(z)$ are polynomials with second-degree, at most, and $\tilde{\tau}(z)$ is a polynomial with first-degree. We define a transformation for the total wave function as

$$\phi(z)=\xi(z)\psi(z). \tag{8}$$

Thus Eq. (7) is reduced to a hypergeometric type equation

$$\sigma(z)\psi''(z)+\tau(z)\psi'(z)+\lambda\psi(z)=0. \tag{9}$$

We also define the new eigenvalue for the Eq. (7) as

$$\lambda\ =\ \lambda_n=-n\tau'-\frac{n(n-1)}{2}\sigma''\,,(n=0,1,2,\ldots) \tag{10}$$

Where

$$\tau(z) \ = \ \tilde{\tau}(z) + 2\pi(z). \tag{11}$$

The derivative of $\tau(z)$ must be negative. $\lambda(\lambda_n)$ is obtained from a particular solution of the polynomial $\psi_n(z)$ with the degree of n. $\psi_n(z)$ is the hypergeometric type function whose solutions are given by [40]

$$\psi_n(z) = \frac{b_n}{\rho(z)} \frac{d^n}{dy^n}[\sigma^n(z)\rho(z)], \tag{12}$$

where the weight function $\rho(z)$ satisfies the equation

$$\frac{d}{dz}[\sigma(z)\rho(z)] = \tau(z)\rho(z). \tag{13}$$

On the other hand, the function $\xi(z)$ satisfies the relation

$$\xi'(z)/\xi(z) = \pi(z)/\sigma(z). \tag{14}$$

Comparing Eq. (6) with Eq. (7), we have

$$\tilde{\tau}(z) = 1 - 2z\,, \quad \sigma(z) = z(1-z)\,, \quad \tilde{\sigma}(z) = -a_1^2 z^2 - a_2^2 z - a_3^2 \tag{15}$$

It becomes

$$\pi(z) = \frac{\sigma'(z) - \tilde{\tau}(z)}{2} \pm \sqrt{(\frac{\sigma'(z) - \tilde{\tau}(z)}{2})^2 - \tilde{\sigma}(z) + k\sigma(z)}\,, \tag{16}$$

or, explicitly

$$\pi(z) = \mp\sqrt{(a_1^2 - k)z^2 + (a_2^2 + k)z + a_3^2}\,, \tag{17}$$

The constant k is determined by imposing a condition such that the discriminant under the square root should be zero. The roots of k are $k_{1,2} = -a_2^2 - 2a_3^2 \mp 2a_3 A$, where $A = \sqrt{a_3^2 + a_2^2 + a_1^2}$. Substituting these values into Eq.(16), we get for $\pi(z)$

$$\pi(z) = \pm \left\{ \begin{array}{l} (A-a_3)z + a_3, \; for\, k_1 = -a_2^2 - 2a_3^2 + 2a_3 A \\ (A+a_3)z + a_3, \; for\, k_2 = -a_2^2 - 2a_3^2 - 2a_3 A \end{array} \right\}. \tag{18}$$

Now we calculate the polynomial $\tau(z)$ from $\pi(z)$ such that its derivative with respect to z must be negative. thus we take the first choice

$$\tau(z) = 1 - 2(1 + A - a_3)z - 2a_3. \tag{19}$$

The constant $\lambda = k + \pi'(z)$ becomes

$$\lambda = -a_2^2 + (2a_3 - 1)(A - a_3)\,, \tag{20}$$

and Eq. (10) gives us

$$\lambda_n = 2n(1 + A - a_3) + n^2 - n\,. \tag{21}$$

Substituting the values of the parameters given by Eq. (5), and setting $\lambda = \lambda_n$, one can find the energy eigenvalues as

$$\begin{aligned} E_n &= V_0(Q^2 m_0 m_1 c^4 - \kappa/2)\tilde{\gamma} \\ &\mp \gamma\tilde{\gamma}\sqrt{4\kappa m_0 m_1 c^4 - (\kappa^2/Q)^2 + 4m_0^2(1/\tilde{\gamma} - Q^2 m_1^2 c^4)} \end{aligned} \tag{22}$$

where

$$\kappa = \frac{1}{2}(2n+1)\left[\frac{1}{2}(2n+1) \mp \sqrt{1+4a_1^2}\right] + \frac{1}{4}\,, \quad \gamma = \sqrt{\kappa + a_1^2}\,, \quad \tilde{\gamma} = \frac{1}{Q^2 V_0^2 + \gamma^2}\,. \tag{23}$$

We see that the energy levels for particles and antiparticles are symmetric about $\frac{V_0(Q^2 m_0 m_1 c^4 - \kappa/2)}{Q^2 V_0^2 + \gamma^2}$. The ground state energy is different from zero. To have a real energy spectra we impose

$$(\kappa^2/2Q)^2 + Q^2 m_0^2 m_1^2 c^4 < \kappa m_0 m_1 c^4 + m_0^2/\tilde{\gamma}\,, \tag{24}$$

We plot four figures to present variation of first three energy eigenvalues as a functions potential parameters V_0 and β. Results are agreement with the ones obtained in the literature [39]. M is the ratio m_1/m_0, and 'p' and 'a' in the brackets represent 'particle'and 'antiparticle' in figures.

Now let us find the eigenfunctions. We first compute the weight function from Eqs. (15) and (19)

$$\rho(z) = z^{-2a_3}(1-z)^{2A}\,, \tag{25}$$

and the wave function becomes

$$\psi_n(z) = \frac{b_n}{z^{-2a_3}(1-z)^{2A}}\,\frac{d^n}{dz^n}\,\left[z^{n-2a_3}\,(1-z)^{n+2A}\right]\,. \tag{26}$$

where b_n is a normalization constant. The polynomial solutions can be written in terms of the Jacobi polynomials [41]

$$\psi_n(z) = b_n\,P_n^{(2A,\,-2a_3)}\,(1-2z)\,,\quad 2A > -1\,,\quad -2a_3 > -1\,. \tag{27}$$

On the other hand, the other part of the wave function is obtained from the Eq.(14) as

$$\xi(z) = z^{a_3}\,(1-z)^{A}\,. \tag{28}$$

Thus, the total eigenfunctions take

$$\phi_n(z) = b'_n\,(1-z)^{A} z^{a_3}\,P_n^{(2A,\,-2a_3)}\,(1-2z) \tag{29}$$

where b'_n is the new normalization constant. It is obtained from

$$\int_0^1 |\phi_n(z)|^2\,dz = 1\,. \tag{30}$$

To evaluate the integral, we use the following representation of the Jacobi polynomials [42]

$$P_n^{(\sigma,\varsigma)}(z) = \frac{\Gamma(n+\sigma+1)}{n!\Gamma(n+\sigma+\varsigma+1)} \sum_{r=0}^{n} \frac{\Gamma(n+1)}{\Gamma(r+1)\Gamma(n-r+1)} \frac{\Gamma(n+\sigma+\varsigma+r+1)}{\Gamma(r+\sigma+1)} (\frac{z-1}{2})^r \,, \tag{31}$$

Hence, from Eq. (30), and with the help of Eq. (31), we get

$$F_{nr}^{2A} \times F_{ms}^{2A} \, |b_n'|^2 \int_0^1 z^{2a_3+r+s} \, (1-z)^{2A} \, dz = 1 \,, \tag{32}$$

where F_{nr}^{2A}, and F_{ms}^{2A} are two arbitrary functions of the parameters A, and a_3, and given by

$$F_{nr}^{2A} = \frac{\Gamma(n+2A+1)}{n!\Gamma(n+2A-2a_3+1)} \sum_{r=0}^{n} \frac{\Gamma(n+1)}{\Gamma(r+1)\Gamma(n-r+1)} \frac{\Gamma(n+2A-2a_3+r+1)}{\Gamma(r+2A+1)} (-1)^r \,, \tag{33}$$

and

$$F_{ms}^{2A} \longrightarrow F_{nr}^{2A} \, (n \longrightarrow m; r \longrightarrow s) \,. \tag{34}$$

The integral in Eq. (32) can be evaluated from the definition of the Beta function [43]

$$B(\mu,\nu) = \int_0^1 y^{\mu-1} \, (1-y)^{\nu-1} \, dy = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} \,, \quad Re(\mu) > 0 \,, \quad Re(\nu) > 0 \,. \tag{35}$$

which gives us

$$\int_0^1 z^{2a_3+r+s} \, (1-z)^{2A} \, dz = \frac{\Gamma(\tilde{\mu}+r+s)\Gamma(\tilde{\nu})}{\Gamma(\tilde{\mu}+\tilde{\nu}+r+s)} \,, \tag{36}$$

where $\tilde{\mu} = 2a_3 + 1$, and $\tilde{\nu} = 2A + 1$.

To get the energy eigenvalues for the constant mass case, we set $q = 1$, and $m_1 = 0$

$$E_n^{m_1=0} = -\frac{V_0}{2} \mp \left[\sqrt{\beta^2 - 4V_0^2} - \beta(2n+1) \right] \left[\frac{m_0^2}{4V_0^2 + [\sqrt{\beta^2 - 4V_0^2} - \beta(2n+1)]^2} - \frac{1}{16} \right]^{1/2} \tag{37}$$

This is the same in Eq. (46) in Ref. [14].

Since the wave function changes only with the parameter A, we simply get

$$\phi_n^{m_1=0}(z) = b_n''\,(1-z)^{A'}\,z^{a_3}\,P_n^{(2A',\,-2a_3)}(1-2z)\,, \tag{38}$$

where the new parameter $A' = \sqrt{a_3^2 - 2\varrho^2 E/V_0 - \varrho^2}$.

B. Non-PT Symmetric and non-Hermitian deformed Woods-Saxon Potential

In this case, we take the potential parameters as $V_0 \to iV_0$, and $\beta \to \beta$. So, the potential takes the form [44]

$$V(x) = -\frac{iV_0}{1+qe^{-\beta x}}\,, \tag{39}$$

From Eq. (3), we obtain

$$\frac{d^2\phi(x)}{dx^2} + \left[\frac{E^2}{\hbar^2c^2} - \frac{m_0^2c^4}{\hbar^2c^2} + \frac{2iEV_0/\hbar^2c^2 - 2m_0m_1c^4/\hbar^2c^2}{(1+qe^{-\beta x})} - \frac{V_0^2/\hbar^2c^2 + m_1^2c^4/\hbar^2c^2}{(1+qe^{-\beta x})^2}\right]\phi(x) = 0\,. \tag{40}$$

By using the same coordinate transformation and defining the following parameters

$$-a_3^2 = Q^2(E^2 - m_0^2c^4)\,, \quad -A_2^2 = Q^2(2iEV_0 - 2m_0m_1c^4)\,, \quad -A_1^2 = Q^2(-m_1^2c^4 - V_0^2)\,, \tag{41}$$

we get

$$\frac{d^2\phi(z)}{dz^2} + \frac{1-2z}{z-z^2}\frac{d\phi(z)}{dz} + \frac{1}{(z-z^2)^2}\left[-A_1^2z^2 - A_2^2z - a_3^2\right]\phi(z) = 0\,. \tag{42}$$

Following the same procedure, we find the energy spectra

$$\begin{aligned} E_n &= \frac{iV_0(2m_0m_1c^4 - \kappa''Q^2)}{2\zeta} \\ &\pm \frac{1}{2\zeta}\sqrt{V_0^2(2m_0m_1 - \kappa''Q^2)^2 - \zeta(\kappa''^2 - 4\kappa''Q^2m_0m_1c^4 + 4m_0^2c^4Q^2(Q^2m_1^2 - \kappa'' - A_1^2))}\,, \end{aligned} \tag{43}$$

where

$$\zeta = Q^2(\kappa'' + Q^2 m_1^2 c^4) \quad , \kappa'' = \frac{1}{2}\,(2n+1)\left[\frac{1}{2}\,(2n+1) + \sqrt{1+4A_1^2}\,\right] + \frac{1}{4}\,. \tag{44}$$

and the corresponding total wave functions as

$$\phi_n(z) = b_n'''\,(1-z)^B\, z^{a_3}\, P_n^{(2B,\,-2a_3)}\,(1-2z)\,, \tag{45}$$

where $B = \sqrt{A_1^2 + A_2^2 + a_3^2}$. It is seen that the energy eigenvalues are consist of the real and imaginary parts. For the constant mass case, we have

$$E_n^{m_1=0} = -\frac{iV_0}{2} \pm \frac{1}{2Q^2\kappa''}\,\sqrt{\kappa''^2 Q^2(\varrho^2 - \kappa'') + 4m_0^2 c^4 Q^4 \kappa''(\kappa'' + \varrho^2)}\,, \tag{46}$$

and the total eigenfunctions are

$$\phi_n^{m_1=0}(z) = b_n''''\,(1-z)^{B'}\, z^{a_3}\, P_n^{(2B',\,-2a_3)}\,(1-2z)\,, \tag{47}$$

where $B' = \sqrt{a_3^2 - 2iE\rho^2/V_0 + \rho^2}$. The energy spectra have real and imaginary part in the constant mass case. The imaginary part does not depend the quantum number n. The normalization constant b_n''' is also obtained in the same way. That is

$$G_{nr}^{2B} \times G_{ms}^{2B}\,|b_n'''|^2 \int_0^1 z^{2a_3+r+s}\,(1-z)^{2B}\,dz = 1\,, \tag{48}$$

The integral can be evaluated by using Eq. (35)

$$\int_0^1 z^{2a_3+r+s}\,(1-z)^{2B}\,dz = \frac{\Gamma(m'+r+s)\Gamma(n')}{\Gamma(m'+n'+r+s)}\,, \tag{49}$$

where $m' = 2a_3 + 1$, and $n' = 2B + 1$. Two functions G_{nr}^{2B}, and G_{ms}^{2B} are given by

$$\begin{aligned} G_{nr}^{2B} &\to F_{nr}^{2A}(2A \to 2B)\,, \\ G_{ms}^{2B} &\to F_{ms}^{2A}(2A \to 2B)\,. \end{aligned} \tag{50}$$

where the functions F_{nr}^{2A}, and F_{ms}^{2A} are defined in Eqs. (33) and (34).

III. CONCLUSION

We have solved the one dimensional effective mass KG-equation for the Woods-Saxon potential. The energy spectra and the corresponding wave functions are obtained by applying the NU-method. We have found a real energy spectra for the WS-potential in the position dependent mass case. To check our results, we have also calculated the energy eigenvalues of the particle and antiparticles for the constant mass limit. We have also studied the non-PT symmetric and non-Hermitian case. We have seen that the energy spectra have real and imaginary parts in this case. We have also obtained the energy spectra and corresponding eigenfunctions for the constant mass limit for this case.

IV. ACKNOWLEDGMENTS

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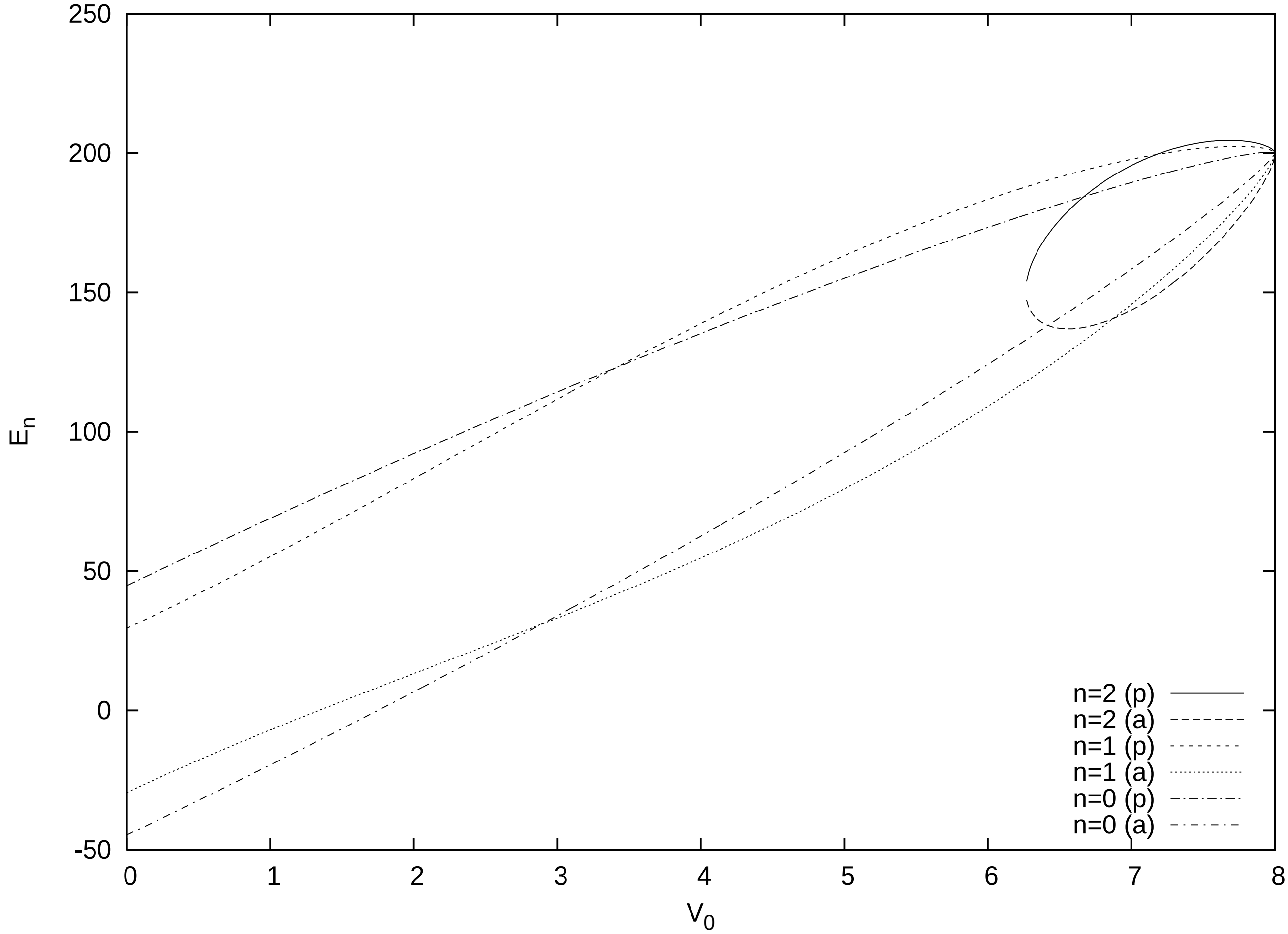

FIG. 1: The dependence of first three excited states on V_0 in the case of $M = 0.04$, and $\beta = 0.1$.

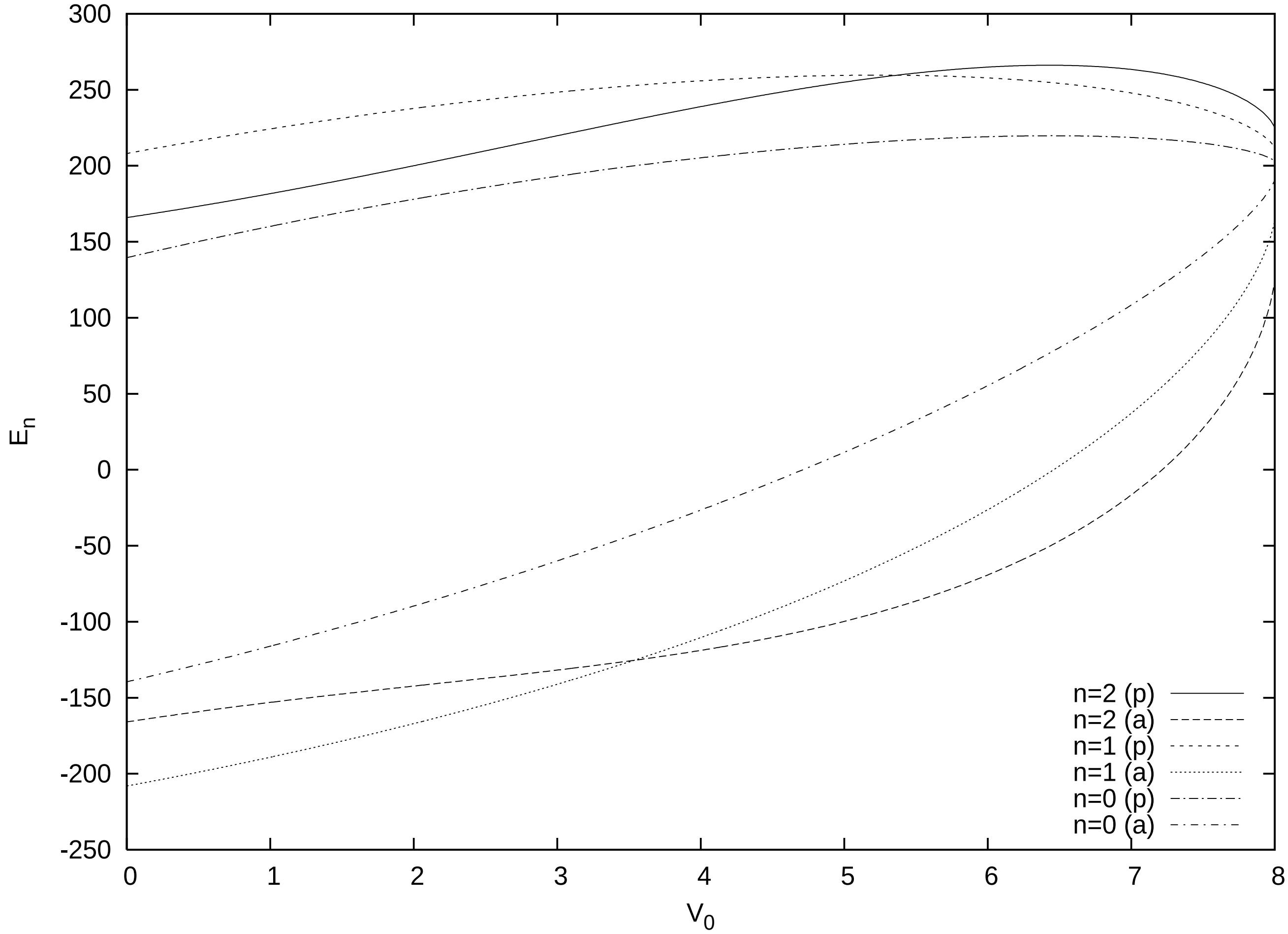

FIG. 2: The dependence of first three excited states on V_0 in the case of $M = 0.04$, and $\beta = 1$.

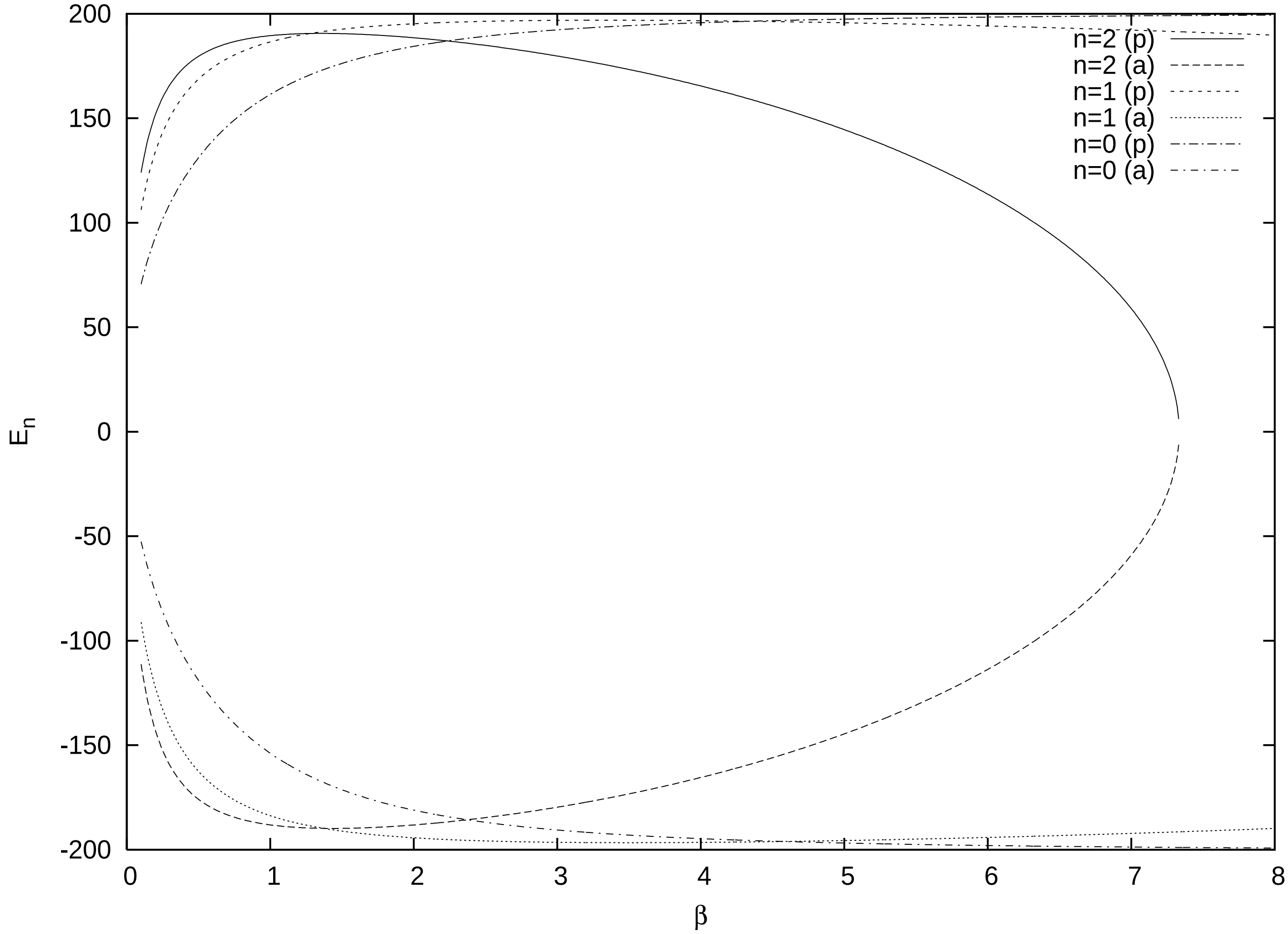

FIG. 3: The dependence of first three excited states on β in the case of $M = 0.01$, and $V_0 = 0.1$.

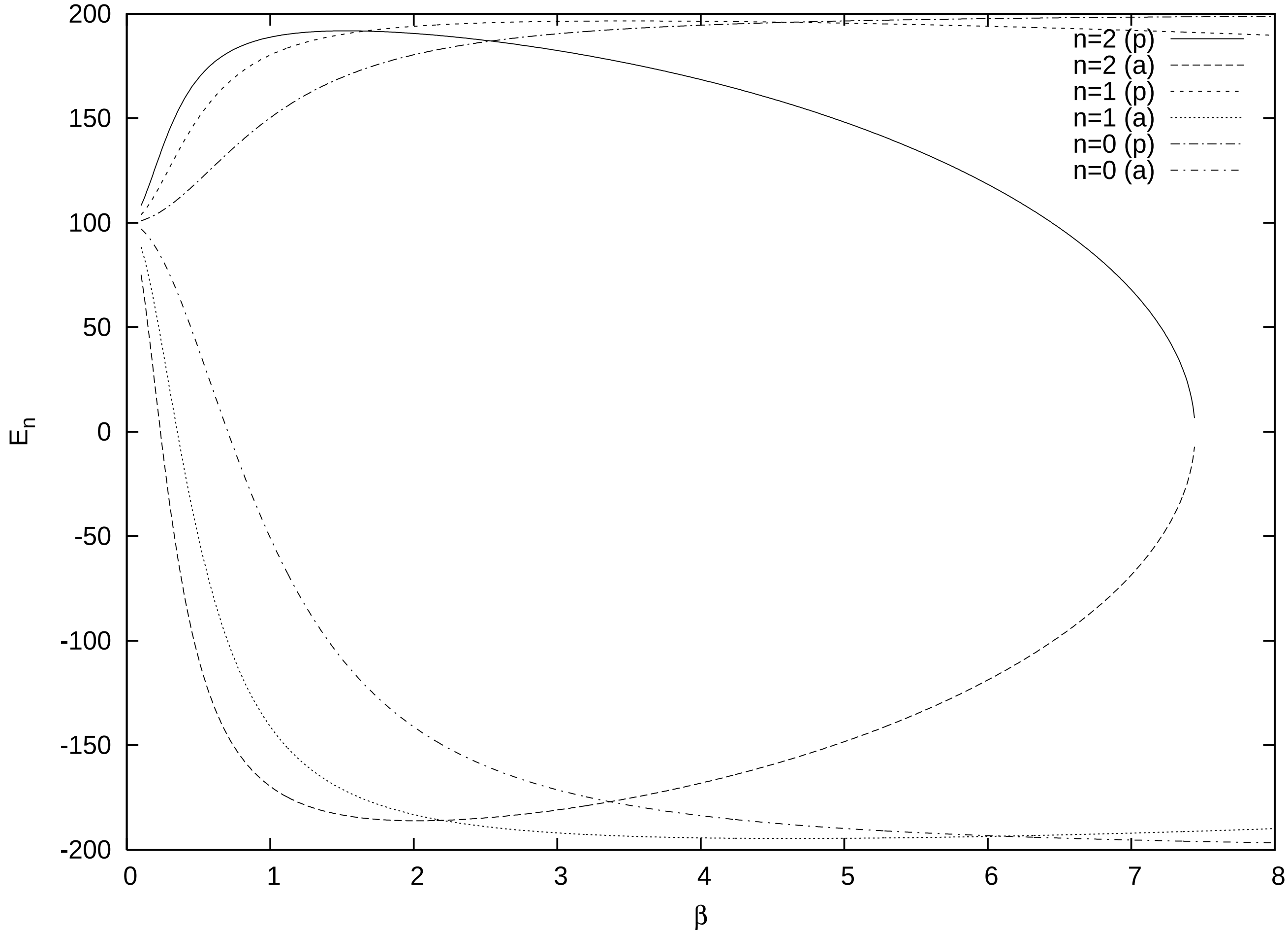

FIG. 4: The dependence of first three excited states on β in the case of $M = 0.01$, and $V_0 = 1$.